

# A SMALL INFINITELY-ENDED 2-KNOT GROUP

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**ABSTRACT.** We show that a 2-knot group discovered in the course of a census of 4-manifolds with small triangulations is an HNN extension with finite base and proper associated subgroups, and has the smallest base among such knot groups.

## 1. INTRODUCTION

Nontrivial classical knot groups have one end. This is equivalent to the asphericity of the knot complement, in the light of Dehn's Lemma and Poincaré duality in the universal cover. In higher dimensions the complements of nontrivial knots are never aspherical. However, Ker-vaire gave a uniform algebraic characterization of  $n$ -knot groups for all  $n \geq 3$ , and a partial characterization of 2-knot groups [8]. (Artin's spinning construction shows that classical knot groups are 2-knot groups and 2-knot groups are high dimensional knot groups.) These characterizations have been used to provide examples of such groups with various properties. In particular, they have been used to find knots whose groups have more than one end.

The 2-twist spins  $\tau_2 K$  of 2-bridge knots  $K$  provide many examples of 2-knot groups with two ends. The first examples of higher dimensional knots with infinitely-ended groups were given in [5]. Their examples are 2-knots, and the groups are HNN extensions with finite base and proper associated subgroups. The simplest of these has presentation

$$\langle a, b, t \mid tat^{-1} = a^2, a^3 = 1, aba^{-1} = b^2 \rangle,$$

with base  $\langle a, b \rangle \cong Z/7Z \rtimes_2 Z/3Z$  and both associated subgroups  $\langle a \rangle \cong Z/3Z$ . (Note that the second and third relations together imply that  $b^7 = 1$ .) It is clear from this presentation that the group is also a free product with amalgamation of  $\langle a, b \rangle$  with  $\langle a, t \rangle \cong \pi\tau_2 3_1$  over  $\langle a \rangle$ , where  $3_1$  is the trefoil knot [9]. In fact it is the group of a satellite of  $\tau_2 3_1$  with companion Fox's Example 10, as is clear from the analysis of the groups of such knots in [7]. Replacing the trefoil with other 2-bridge knots gives all of the examples of §1 of [5].

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1991 *Mathematics Subject Classification.* 57Q45.

*Key words and phrases.* end, 2-knot, 4-manifold, virtually free.

In §2 we shall show that a knot exterior recently discovered in the course of a computational census of 4-manifolds with small triangulations has group  $\pi$  which is an HNN extension with base the generalized quaternionic group  $Q(16)$  and (distinct) associated subgroups  $Q(8)$ . This group is not properly the group of a satellite knot (§3). In §4 we show that no smaller finite group is the base of an HNN extension which is an infinitely-ended knot group, and that there is only one such group with base  $Q(16)$ . However there are such groups with base  $D_8 \times Z/2Z$  and associated subgroups  $(Z/2Z)^3$  (§5). In §6 we show that all knots with the exterior described in §2 are strongly +amphicheiral, and we determine generators for  $Out(\pi) \cong (Z/2Z)^3$ .

For us, an  $n$ -knot is a locally flat embedding of  $S^n$  in  $S^{n+2}$ , and so orientations of the spheres determine orientations of the knot complements and preferred meridians for the knots. We shall use Chapter 14 of [6] as a one-stop reference for the aspects of higher dimensional knot theory that we need. All homology groups considered below shall have integral coefficients, and so we shall write  $H_i(G)$  instead of  $H_i(G; \mathbb{Z})$ .

## 2. THE KNOT EXTERIOR

The knot exterior of this paper was discovered while forming a census of 4-manifolds triangulable with 6 or less pentachora. Precisely, Ben Burton generalized the census-generation algorithm in Regina [2] to enumerate all unordered 4-dimensional delta-complexes whose vertex links are triangulated 3-spheres or more generally 3-manifolds. Triangulations having a non-spherical manifold vertex link are called *cusped triangulated manifolds*. Most of the non-trivial knot exteriors in the census are of ideal/cusped type. A previous paper was written on the simplest non-trivial knot exterior in the census [1]. A future paper will describe the census in full.

In the census there are approximately 1.4 million combinatorial classes of knot exteriors in homotopy spheres. By *combinatorial class* we mean triangulated manifolds, up to homeomorphism that preserve the simplicial subdivision from the triangulation, i.e. the homeomorphisms need not preserve the characteristic maps of the individual simplices. Among these 1.4 million triangulations, 8521 have non-abelian fundamental group. Most of these have finitely generated commutator subgroup. There are just twenty exceptional cases.

Ten of these have group  $\Phi$  with presentation  $\langle a, t \mid tat^{-1} = a^2 \rangle$ , and so are homeomorphic to the exterior of Fox's Example 10. The final ten all have fundamental group isomorphic to the group  $\pi$  with

presentation

$$\langle a, t \mid a^8, a^{-2}tata^2t^{-2}, a^2tata^{-2}t^{-2}, a^4t^{-1}a^{-4}t \rangle.$$

We briefly describe one of the ten triangulations giving a manifold  $M$  with fundamental group  $\pi$ . At present we know there are at most two PL homeomorphism types represented by these triangulations.

M	(0123)	(0124)	(0134)	(0234)	(1234)
0	4 (0123)	3 (0124)	2 (1320)	2 (0234)	1 (1234)
1	5 (0123)	4 (1240)	4 (4320)	2 (1234)	0 (1234)
2	0 (4031)	5 (3142)	3 (4013)	0 (0234)	1 (0234)
3	5 (0421)	0 (0124)	2 (1340)	4 (1423)	5 (4031)
4	0 (0123)	1 (4012)	5 (0324)	1 (4310)	3 (0342)
5	1 (0123)	3 (0321)	3 (2431)	4 (0314)	2 (1402)

The leftmost column lists the pentachora of the triangulation, labelling them 0 through 5. In each row, to the right of the pentachoron index is a collection of pairs  $n(abcd)$ . The first number  $n$  indicates on which pentachoron this tetrahedral facet is glued to. The entry  $(abcd)$  indicates the affine-linear map on the tetrahedral facet. For example, the row 1 5(0123) 4(2104) 4(4320) 2(1234) 0(1234) indicates that in the pentachoron indexed by 1, the 4th tetrahedron is glued to the 4th tetrahedron but in pentachoron 5, with vertices (0123) glued to (0123) in that order. Similarly, the 2nd tetrahedron is glued to the 1st tetrahedron but in pentachoron 4, with vertices (0134) sent to (4320) in that order, etc. We leave the readers to consult the documentation for [2] for details. In summary, this triangulation has no internal vertices – after performing the above gluings, all the vertices have been identified, thus the single vertex has vertex link a 30-tetrahedron triangulation of  $S^1 \times S^2$ . The triangulation has 3 edges, 12 triangles, 15 tetrahedra and 6 pentachora.

Verification that  $M$  is a knot exterior in a homotopy sphere is similar to the argument in [1] and left to the reader. (Up to changes of orientation, there are at most two knots with exterior homeomorphic to  $M$ .) Budney and Burton have automated the process and it is implemented in the software [2]. Perhaps for some readers it would be more appealing to read the algorithm implemented in Regina. At present the 4-manifolds tools are in the *development repository* of Regina, and these tools will be in the general release of Regina by version 5.0. One can readily check that the above triangulation has a single non-trivial symmetry, an involution that reverses orientation and acts non-trivially on  $H_1(M)$ . The involution is the simplicial map that sends pentachoron

0 to pentachoron 2, sending vertices (01234) to (24103), in that order. Pentachoron 1 is sent to pentachoron 3, sending vertices (01234) to (23041), in that order, and pentachoron 4 to pentachoron 5, sending vertices (01234) to (42130), in that order.

Returning to the group  $\pi = \pi_1(M)$ , we find that the third relator in the above presentation is a consequence of the others, and so this presentation simplifies to

$$\langle a, t \mid tat^{-1} = a^2t^2a^{-2}t^{-2}, a^8, a^4t = ta^4 \rangle.$$

Setting  $b = ta^2t^{-1}$ , this becomes

$$\langle a, b, t \mid ta^2t^{-1} = b, tabt^{-1} = a^2, a^4 = b^2 = (ab)^2 \rangle.$$

(The final two relations imply that  $bab^{-1} = a^{-1}$ . Hence  $a^8 = 1$ , and so  $a^4$  is a central involution.) Thus  $\pi \cong B *_H \varphi$  is an HNN extension, with base  $B \cong Q(16)$ , the generalized quaternionic group with presentation

$$\langle a, b \mid a^4 = b^2 = (ab)^2 \rangle.$$

and associated subgroups  $H = \langle a^2, ab \rangle \cong \varphi(H) = \langle a^2, b \rangle \cong Q(8)$ . The commutator subgroup is an iterated generalized free product with amalgamation

$$\pi' \cong \dots B *_H B *_H B \dots,$$

and is perfect ( $\pi' = \pi''$ ). Since  $H$  and  $\varphi(H)$  are proper subgroups of  $B$  the commutator subgroup is not finitely generated. Hence no knot with this group is fibred.

### 3. $\pi$ IS NOT PROPERLY THE GROUP OF A SATELLITE KNOT

If an  $n$ -knot  $K$  is a satellite of  $K_1$  about  $K_2$  relative to a simple closed curve  $\gamma$  in  $X(K_1)$  then

$$\pi K \cong \pi K_2 / \langle \langle w^q \rangle \rangle *_{w=[\gamma]} \pi K_1,$$

where  $[\gamma] \in \pi K_1$  has order  $q \geq 0$  and  $w$  is a meridian for  $K_2$ . The case  $q = 0$  corresponds to  $[\gamma]$  having infinite order. (See [7], or page 271 of [6].) If  $K_2 = \tau_r k$  is the  $r$ -twist spin of an  $(n-1)$ -knot  $k$  and  $(q, r) = 1$  then  $\pi K_2 / \langle \langle w^q \rangle \rangle \cong Z/qZ$ . Thus every 2-knot group with non-trivial torsion is trivially the group of a satellite knot. We shall say that  $\pi K$  is *properly* the group of a satellite knot if  $|\pi K_2 / \langle \langle w^q \rangle \rangle| > q$ .

Suppose that the group  $\pi$  of §2 is properly the group of a satellite knot. Since  $\pi$  has a central subgroup of order 2, it is not of the form  $A *_Z B$  with  $A$  and  $B$  nontrivial knot groups. Hence  $\pi \cong G *_Z H$ , where  $G$  is a knot group and  $H$  is the quotient of a knot group by the  $q$ th power of a meridian, for some  $q > 0$ , but is not cyclic.

Since  $\pi$  is an HNN extension with finite base it is virtually free. (See Corollary IV.1.9 of [4]. The argument given there implies that  $\pi$  has a free subgroup of index dividing  $16!$ .) Therefore so are  $G$  and  $H$ , and all these groups have well-defined virtual Euler characteristics. Mayer-Vietoris arguments give

$$\chi^v(\pi) = \chi^v(Q(16)) - \chi^v(Q(8)) = \frac{1}{16} - \frac{1}{8} = -\frac{1}{16}$$

and

$$\chi^v(\pi) = \chi^v(G) + \chi^v(H) - \frac{1}{q},$$

while  $\chi^v(G) \leq 0$ , since  $G$  is an infinite virtually free group. Hence

$$\chi^v(H) \geq \frac{1}{q} - \frac{1}{16}.$$

Now since  $\pi \cong Q(16) *_{Q(8)} \varphi$ , any finite subgroup of  $\pi$  is conjugate to a subgroup of  $Q(16)$ . Therefore  $q$  divides 8, and so  $\chi^v(H) > 0$ . Hence  $H$  is finite, and so it is isomorphic to a subgroup of  $Q(16)$ .

We then find that the only possibility is that  $q = 8$  and  $H \cong Q(16)$ . But then  $\chi^v(G) = 0$ , and so  $G'$  is finite, and is either  $Z/8Z$  or  $Q(16)$ . Neither of these groups admits a meridional automorphism, and so there is no such knot group  $G$ . Thus we may conclude that  $\pi$  is not properly the group of a satellite knot.

HNN extensions arise naturally in knot theory when an  $n$ -knot  $K$  has a minimal Seifert hypersurface  $V$ , one for which the pushoffs from  $V$  to either side of  $Y = S^{n+2} \setminus V$  are both  $\pi_1$ -injective. The knot group  $\pi K$  is then an HNN extension with base  $\pi_1(Y)$  and associated subgroups isomorphic to  $\pi_1(V)$ . There are 2-knots with group  $Z/3Z \rtimes \mathbb{Z}$  (the group of the 2-twist spun trefoil) which do not have minimal Seifert hypersurfaces. (See Chapter 17 of [6].) Does a 2-knot with exterior the manifold  $M$  of §2 have a minimal Seifert hypersurface realizing the HNN structure  $\pi \cong Q(16) *_{Q(8)} \varphi$ ? (Knots related by composition with reflections of  $S^n$  or  $S^{n+2}$  have similar Seifert hypersurfaces.)

#### 4. HNN EXTENSIONS WITH SMALL FINITE BASE

Let  $G = B *_C \phi$  be an HNN extension with base  $B$  and associated subgroups  $C$  and  $\phi(C)$ . Let  $j : C \rightarrow B$  be the natural inclusion. Consideration of the Mayer-Vietoris sequence for the extension shows that  $H_1(G) \cong \mathbb{Z}$  and  $H_2(G) = 0$  if and only if  $H_1(j) - H_1(\phi)$  is an isomorphism and  $H_2(j) - H_2(\phi)$  is surjective. If the  $H_1$  condition holds then  $B/N$  is perfect, where  $N = \langle\langle \{j(g)^{-1}\phi(g) | g \in C\} \rangle\rangle$  is the normal closure of  $\{j(g)^{-1}\phi(g) | g \in C\}$  in  $B$ . In particular, if  $B$  is solvable then

$N = B$ . The  $H_2$  condition holds automatically if  $H_2(B; \mathbb{Z}) = 0$ , in particular, if  $B$  is a finite group of cohomological period 4.

If both homological conditions hold and  $N = B$  then the stable letter of the HNN extension normally generates  $G$ , and so  $G$  is a knot group [5]. (However, such HNN extensions need not be 2-knot groups. The group  $Z/5Z \rtimes_2 Z$  is a high dimensional knot group which is an HNN extension with  $H = B = Z/5Z$ , but the Farber-Levine condition fails, since  $2^2 \not\equiv 1 \pmod{5}$ . See Chapter 14 of [6].)

In particular, if  $B$  is finite and  $C$  is a proper subgroup then  $B$  is nonabelian, and  $C/C' \cong B/B'$ . Hence  $H_1(B)$  cannot be cyclic of even order. Therefore  $B$  is neither a dihedral group  $D_{2k}$  with  $k$  odd, nor  $Z/3Z \rtimes_{-1} Z/4Z$ . This leaves only  $Q(8)$ ,  $D_8$ ,  $D_{12}$  and  $A_4$  among the groups of order less than 16.

The group  $Q(8)$  has no proper subgroup with abelianization  $(Z/2Z)^2$ . If  $B = D_{4k}$  then  $C$  must be isomorphic to  $D_{4l}$ , for some  $l$  dividing  $k$ . But then  $H_2(C) \cong H_2(B) = Z/2Z$ , and  $H_2(j)$  and  $H_2(\phi)$  are the same isomorphism. Hence  $H_2(j) - H_2(\phi)$  is not an epimorphism. Thus we may exclude  $D_8$  and  $D_{12}$ . If  $B = A_4$  then  $C$  must be  $Z/3Z$ . Since  $H_2(Z/3Z) = 0$  and  $H_2(A_4) = Z/2Z$ , this group may be excluded also.

Thus the smallest possible base must have order at least 16. The group  $Q(16)$  has two proper subgroups with abelianization  $(Z/2Z)^2$ . These are  $\langle a^2, b \rangle$  and  $\langle a^2, ab \rangle$ , and are isomorphic to  $Q(8)$ . The automorphism  $a \mapsto a, b \mapsto ab$  of  $Q(16)$  carries one onto the other. Fix generators  $x, y$  for  $Q(8)$ . Then we may assume that  $j(x) = a^2$  and  $j(y) = b$ . If  $\phi$  is another embedding then  $\phi(x)$  and  $\phi(y)$  have order 4, so one must be  $a^{\pm 2}$  and the other  $a^i b$ . If, moreover,  $H_1(j) - H_1(\phi)$  is an isomorphism then  $\phi(x) = a^i b$  and  $\phi(y) = a^{\pm 2}$ , and  $i$  must be odd. After conjugation in  $Q(16)$  we may assume that  $\phi(x) = ab$  and  $\phi(y) = a^2$ . Since  $Q(16)$  is solvable and  $H_2(Q(16)) = 0$  the HNN extension  $\pi = Q(16) *_{Q(8)} \varphi$  is a knot group, and it has the smallest finite base among all such HNN extensions. Moreover, it is the unique such group with HNN base  $Q(16)$ .

## 5. FURTHER EXAMPLES WITH BASE OF ORDER 16

There are eight other non-abelian groups of order 16. Four are semidirect products  $K \rtimes L$  with  $K$  and  $L$  cyclic. In three of these four cases  $H_2(C) = H_2(B) = Z/2Z$ , and so these may be ruled out, by the argument that excluded  $D_8$  and  $D_{12}$ . The fourth group  $M_{16}$  has presentation  $\langle a, x \mid a^8 = x^2 = 1, xax = a^5 \rangle$ . The abelianization is  $Z/4Z \oplus Z/2Z$ , and so  $C$  must be  $\langle a^2, x \rangle$ . There is no second embedding

$\phi$  such that  $H_1(j) - H_1(\phi)$  is an isomorphism, and so we may rule out  $M_{16}$ . (Note that  $H_2(M_{16}) = 0$ .)

The next group to consider is  $(Z/2Z)^2 \rtimes_{\theta} Z/4Z$ , the semidirect product with action generated by  $\theta = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in GL(2, \mathbb{F}_2)$ , and with presentation  $\langle a, b, x \mid a^4 = b^2 = x^2 = 1, ab = ba, bx = xb, axa^{-1} = bx \rangle$ . (Since  $b = a^2(ax)^2$ , this group is generated by  $\{a, x\}$ .) This has abelianization  $Z/4Z \oplus Z/2Z$ , and so  $H_2(C) = Z/2Z$ . It follows from the LHS spectral sequence for  $B$  as a semidirect product that  $H_2(B)$  maps onto  $H_1(Z/4Z; (Z/2Z)^2) = \text{Ker}(\theta - I) = Z/2Z$ . Hence either  $H_2(B) = Z/2Z$  and  $H_2(j) - H_2(\phi) = 0$ , or  $H_2(B)$  has order  $\geq 4$ . In neither case is  $H_2(j) - H_2(\phi)$  an epimorphism, and so we may exclude this group.

The remaining three have abelianization  $(Z/2Z)^3$ , so  $C$  must be an abelian subgroup of index 2, and hence normal. These are  $Q(8) \times Z/2Z$ ,  $D_8 \times Z/2Z$  and the central product of  $D_8$  with  $Z/4Z$ , with presentation

$$\langle a, c, x \mid a^4 = x^2 = 1, a^2 = c^2, ac = ca, cx = xc, xax = a^{-1} \rangle.$$

We may eliminate  $Q(8) \times Z/2Z$  and the central product immediately, as neither has a proper subgroup with abelianization  $(Z/2Z)^3$ .

The final group is  $B = D_8 \times Z/2Z$ , with presentation

$$\langle a, b, x \mid a^4 = b^2 = x^2 = 1, ab = ba, bx = xb, xax = a^{-1} \rangle.$$

There are two proper subgroups isomorphic to  $(Z/2Z)^3$ . These are  $\langle a^2, b, x \rangle$  and  $\langle a^2, b, ax \rangle$ , and the automorphism  $a \mapsto a, b \mapsto b, x \mapsto ax$  of  $B$  carries one onto the other. Let  $\{c_1, c_2, c_3\}$  be the standard basis for  $(Z/2Z)^3$ . Then we may assume that  $C$  is the image of the embedding  $j : (Z/2Z)^3 \rightarrow B$  determined by  $j(c_1) = a^2, j(c_2) = b$  and  $j(c_3) = x$ .

Let  $V = (Z/2Z)^2$ , and let  $e_i$  be the image of the generator of  $H_2(V) = Z/2Z$  under the inclusion of  $V$  onto the subgroup generated by  $\{c_k \mid k \neq i\}$ . Then  $\{e_1, e_2, e_3\}$  is a basis for  $H_2(C) \cong (Z/2Z)^3$ . We also have  $H_2(B) \cong (Z/2Z)^3$ , since  $H_2(B) = H_2(D_8) \oplus (H_1(D_8) \otimes Z/2Z)$ , by the Künneth Theorem. This has a basis  $\{f_1, f_2, f_3\}$ , where  $f_1$  is the image of the generator of  $H_2(D_8)$ ,  $f_2 = a \otimes b$  and  $f_3 = x \otimes b$ . The homomorphism  $H_2(j)$  sends  $e_1, e_2$  and  $e_3$  to  $f_3, f_1$  and 0, respectively.

Reimbeddings satisfying the  $H_1$  condition must carry  $C$  to  $\tilde{C} = \langle a^2, b, ax \rangle$ . There are  $|GL(3, \mathbb{F}_2)| = 168$  possible isomorphisms  $\phi$ . Conjugation in  $B$  reduces this by half (since  $[B : C] = 2$ ), but this still leaves too many possibilities to examine easily by hand. We shall just give one example.

Let  $\tilde{j}(c_1) = ax, \tilde{j}(c_2) = a^2$  and  $\tilde{j}(c_3) = b$ . Then  $\text{Im}(\tilde{j}) = \tilde{C}$ , and  $H_2(\tilde{j})$  sends  $e_1, e_2$  and  $e_3$  to 0,  $f_2 + f_3$  and  $f_1$ , respectively. It follows easily that the homological conditions are satisfied. Let  $\phi = \tilde{j}^{-1}$  (so

$\phi(a^2) = ax$ ,  $\phi(b) = a^2$  and  $\phi(x) = b$ , and let  $\pi = B *_C \phi$ . Then the stable letter of the HNN extension is a normal generator for  $\pi$ , since  $B$  is solvable, and so  $\pi$  is a high-dimensional knot group. Is there a 2-knot group with HNN basis  $B = D_8 \times Z/2Z$ ?

## 6. SYMMETRIES

Four of the ten (ideal) triangulations in the census which realize  $\pi$  have simplicial involutions which reverse orientation and acts non-trivially on  $H_1(M)$ . In particular, the triangulation displayed in §2 has a single non-trivial symmetry, which is such an involution. It is the simplicial map that sends pentachoron 0 to pentachoron 2, sending vertices (01234) to (24103), in that order. Pentachoron 1 is sent to pentachoron 3, sending vertices (01234) to (23041), in that order, and pentachoron 4 to pentachoron 5, sending vertices (01234) to (42130), in that order.

The open 4-manifold  $M$  is the interior of a compact 4-manifold  $\overline{M}$ , with boundary  $\partial\overline{M} \cong S^1 \times S^2$ , and the involution extends to  $\overline{M}$ . It is known that there are 13 involutions of  $S^1 \times S^2$ , up to conjugacy [10]. In Tollefson's classification precisely three reverse both the orientation and the meridian, and they are determined by their fixed-point sets. One has fixed-point set  $S^0 \amalg S^2$ , and does not extend across  $D^2 \times S^2$ . The others have fixed-point set  $S^2 \amalg S^2$  and  $S^0 \amalg S^0$ , respectively, and extend to involutions of  $D^2 \times S^2$ . Computation shows that the present involution fixes  $S^0 \amalg S^0$  (i.e., four points) in  $\partial\overline{M}$ , and thus extends across any homotopy 4-sphere of the form  $M \cup D^2 \times S^2$ . Hence every knot with exterior  $M$  is strongly +amphicheiral. Is any such knot also invertible or reflexive?

On the algebraic side, it is easy to determine the outer automorphism classes of  $\pi$ , since every automorphism of an HNN extension with finite base must carry the base to a conjugate of itself. Thus  $Out(\pi)$  is generated by automorphisms which fix  $Q(16)$  set-wise. If  $\alpha$  is such an automorphism then  $\alpha(a) = a^i$  and  $\alpha(b) = a^j b$ , for some odd  $i = \pm 1$  and some  $j$ , and  $\alpha(t) = wt^\epsilon$ , for some  $w \in \pi' = \langle\langle a \rangle\rangle$  and  $\epsilon = \pm 1$ .

Suppose first that  $\epsilon = 1$ . The images of the relations give equations

$$wb^i w^{-1} = a^j b \quad \text{and} \quad wta^{i+j}bt^{-1}w^{-1} = a^{2i}.$$

The first equation implies that  $w \in Q(16)$ , by the uniqueness of normal forms for elements of an HNN extension. Hence  $j$  must be even, and so  $i + j = 2k + 1$ , for some  $k$ . The second equation then becomes

$$wb^k a^2 w^{-1} = a^{2i},$$



and so  $k$  is also even. On following this through, we find that there is an unique such automorphism for each  $w \in Q(16)$ . Four of these are inner automorphisms, given by conjugation by elements of the subgroup  $\langle a^2, ab \rangle$ , and so we need only consider the automorphisms  $f$  and  $g$ , given by  $f(a) = a$ ,  $f(b) = b$  and  $f(t) = a^4t$ , and  $g(a) = a^{-1}$ ,  $g(b) = a^{-2}b$  and  $g(t) = at$ . It is easy to see that  $fg = gf$  and  $f^2 = g^2 = id_\pi$ .

There is also an automorphism  $h$  such that  $h(a) = a$ ,  $h(b) = ab$  and  $h(t) = (at)^{-1}$ . This automorphism induces the involution of  $\pi/\pi' \cong \mathbb{Z}$ . We have  $fh = hf$  and  $(gh)^2 = id_\pi$ , while  $h^2$  is conjugation by  $a$ , and so  $h^8 = id_\pi$ . Thus  $Out(\pi) \cong (Z/2Z)^3$ , and is generated by the images of  $f$ ,  $g$  and  $h$ .

The HNN structure determines a Mayer-Vietoris sequence

$$\cdots \rightarrow H_4(Q(16)) \rightarrow H_4(\pi) \rightarrow H_3(Q(8)) \rightarrow H_3(Q(16)) \rightarrow \cdots,$$

where the right hand homomorphism is the difference of the homomorphisms induced by the inclusions of the two associated subgroups. Since  $H_3(Q(8)) \cong Z/8Z$ ,  $H_3(Q(16)) \cong Z/16Z$  and  $H_4(Q(16)) = 0$ , it follows that  $H_4(\pi)$  is cyclic of order dividing 8. Since both of the homomorphisms  $H_3(Q(8)) \rightarrow H_3(Q(16))$  induced by the inclusions are injective,  $H_3(\pi) \neq 0$ . The automorphisms  $f$  and  $g$  preserve the associated subgroups, and induce the identity on  $H_3(Q(8))$ . Hence they also induce the identity on  $H_4(\pi)$ . How does  $h$  act on this homology group?

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